

# COLLISIONLESS MODES OF A TRAPPED BOSE GAS

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## Abstract

We calculate the excitation frequencies of the  $m = 0$  monopole and  $m = 2$  quadrupole modes in the collisionless regime by solving a non-linear Schrödinger equation for the condensate, coupled to a collisionless Boltzmann equation for the quasiparticles. Since the dynamics of the non-condensate cloud is also taken into account, the theory satisfies the Kohn theorem. The spectrum turns out to be strongly temperature dependent and we compare our results with experiment.

## 1 INTRODUCTION

Since the experimental realization of a Bose condensate in trapped alkali vapors, there has been renewed interest into the subject of degenerate quantum gases. Topics like the equilibrium properties of the condensate, the dynamics of condensate formation, topological defects, and collective excitations have been studied extensively. Of particular interest are the collective excitations in the collisionless regime, because in this regime, where the mean free path of the Bogoliubov quasiparticles is much larger than the wavelength of the collective excitations, there seems to be a discrepancy between the experimental observations and theoretical calculations. At temperatures far below the critical temperature  $T_c$ , measurements of the low-lying collective excitations [1, 2] are in excellent agreement with theoretical calculations solving the Gross-Pitaevskii equation [3, 4, 5, 6], which describes the condensate dynamics at zero temperature. At higher temperatures there is a considerable noncondensate fraction, and one has to include the mean-field interaction of the thermal cloud into the evolution equation for the condensate wave function. Theoretical calculations solving the resulting nonlinear Schrödinger equation predict almost no temperature dependence of the lowest excitation frequencies [7, 8], whereas experiments clearly show a large temperature dependence [9]. This might partly be explained by including in the effective interaction between two colliding particles the many-body effect of the surrounding gas on the collisions, which causes the effective two-particle interaction to become strongly temperature dependent [10]. The frequencies of the low-lying modes will therefore also depend on temperature

[11]. However, in these approaches the nonlinear Schrödinger equation describes the dynamics of the condensate in the presence of a static noncondensed cloud. As a result they violate the Kohn theorem, which states that there should always be three center-of-mass modes with the trapping frequencies. Clearly, this violation is caused by the fact that we also have to describe the time evolution of the thermal cloud. Hence we propose to describe the collective excitations in the collisionless regime by a nonlinear Schrödinger equation for the condensate wave function that is coupled to a collisionless Boltzmann equation describing the dynamics of the noncondensed atoms [12, 13]. This resolves our problem, because the resulting theory can be shown to contain the Kohn modes exactly. A full solution of the collisionless Boltzmann equation for the distribution function of the Bogoliubov quasiparticles is rather complicated. Therefore we apply as a first step in this article the Hartree-Fock approximation for the quasiparticle dispersion. This is appropriate in the most interesting region, near the critical temperature  $T_c$ , where the mean-field interaction of the condensate is small compared to the average kinetic energy of the noncondensed cloud. We then determine the eigenfrequencies of the low-lying modes by a variational approach and compare these to the experiments.

## 2 COLLISIONLESS DYNAMICS

Our aim is to describe the coupled dynamics of the condensate and the thermal cloud in the collisionless regime, where the wavelength of the collective excitations is much larger than the mean free path of the quasiparticles. The time evolution of the condensate wave function  $\Phi(\mathbf{x}, t)$  describes the dynamics of both the condensate density  $n_0(\mathbf{x}, t) = |\Phi(\mathbf{x}, t)|^2$ , and the superfluid velocity  $\mathbf{v}_s(\mathbf{x}, t) = \hbar [\Phi^*(\mathbf{x}, t) \nabla \Phi(\mathbf{x}, t) - \Phi(\mathbf{x}, t) \nabla \Phi^*(\mathbf{x}, t)] / 2imn_0(\mathbf{x}, t)$ . In the mean-field approximation it obeys the nonlinear Schrödinger equation, that includes the effect of the mean-field interaction of the noncondensate density  $n'(\mathbf{x}, t)$ , and reads

$$i\hbar \frac{\partial \Phi(\mathbf{x}, t)}{\partial t} = \left\{ -\frac{\hbar^2 \nabla^2}{2m} + V^{trap}(\mathbf{x}) - \mu + T^{2B} [2n'(\mathbf{x}, t) + n_0(\mathbf{x}, t)] \right\} \Phi(\mathbf{x}, t)$$

Here,  $V^{trap}(\mathbf{x})$  denotes the external trapping potential. The factor of two difference between the condensate and noncondensate mean-field contributions results from the symmetrisation of the many-body wave function. For the noncondensate part this contributes a Hartree and a Fock term, but for the condensate part only a Hartree term. Moreover, the Hartree and the Fock contributions are equal because the two-body interaction is approximated by a hard-core potential  $V(\mathbf{x} - \mathbf{x}') = T^{2B} \delta(\mathbf{x} - \mathbf{x}')$ , where the two-body scattering matrix  $T^{2B} = 4\pi\hbar^2 a/m$  solves the Lipmann-Schwinger equation for the scattering of two particles with zero momentum. In the Thomas-Fermi limit one can neglect the average kinetic energy of the condensate relative to the mean-field interactions, and the equilibrium density profile of the condensate is approximately an inverted parabola. In the opposite limit where the mean-field interaction is

much less than the average kinetic energy, the result is essentially a gaussian density profile associated with the ground state of an harmonic oscillator.

The Boltzmann equation is obtained by performing a gradient expansion on the equations of motion for the Wigner distribution, i.e. the Fourier transform of the one-particle density matrix. This is justified because the noncondensate density profile varies on a much larger length scale than the external trapping potential. It describes the time evolution of the quasiparticle distribution function  $F(\mathbf{k}, \mathbf{x}, t)$ , which gives the noncondensate density profile, basically by integrating over momentum space. The local group velocity and the local force on a quasiparticle are given by the momentum and the spatial derivative of the dispersion, respectively. Therefore the Boltzmann equation reads

$$\left[ \frac{\partial}{\partial t} + \frac{\partial \omega(\mathbf{k}, \mathbf{x}, t)}{\partial \mathbf{k}} \cdot \frac{\partial}{\partial \mathbf{x}} - \frac{\partial \omega(\mathbf{k}, \mathbf{x}, t)}{\partial \mathbf{x}} \cdot \frac{\partial}{\partial \mathbf{k}} \right] F(\mathbf{k}, \mathbf{x}, t) = \left[ \frac{\partial F(\mathbf{k}, \mathbf{x}, t)}{\partial t} \right]_{collisions} \quad (2)$$

Sufficiently close to the critical temperature, where  $k_B T \gg T^{2B} n_0(\mathbf{x})$ , we can treat the quasiparticles in the Hartree-Fock approximation and the dispersion is accurately given by

$$\omega(\mathbf{k}, \mathbf{x}, t) = \frac{\hbar^2 \mathbf{k}^2}{2m} + V^{trap}(\mathbf{x}) + 2T^{2B} [n'(\mathbf{x}, t) + n_0(\mathbf{x}, t)] . \quad (3)$$

The equilibrium distribution can be found by requiring the collision term to be equal to zero. For the equilibrium density profile the result is equivalent to a local density approximation. This is accurate for the conditions of interest, since also the coherence length of the gas is much smaller than the the scale on which the equilibrium profile varies. To find the collisionless excitation spectrum, we now consider fluctuations around equilibrium. Since the eigenfrequencies of the lowest collisionless excitations are much larger than the average time between two collisions, we can neglect the contribution from the collisional term. The result is thus a linearization of the collisionless Boltzmann equation, which is also known in the context of plasma physics as the Vlasov equation.

### 3 VARIATIONAL APPROACH

Variational calculations have been quite succesfull in obtaining the frequencies of the low-lying modes at zero temperature. Because of this, and because solving the coupled equations (1) and (2) is rather difficult, we use a variational method to solve these equations. Therefore we need two variational functions. One for the condensate wave function, the other for the quasiparticle distribution. Here, the external trapping potential is a quadratic potential  $V^{trap}(\mathbf{x}) = \sum_i m \omega_i^2 x_i^2 / 2$ . In the experiments performed at JILA and MIT the trap is cylindrically symmetric, hence  $\omega_1 = \omega_2 = \omega_r$ , and  $\omega_3 = \omega_z$ .

First, the condensate wave function is approximated by a gaussian,

$$\Phi(\mathbf{x}, t) = N_0 \left[ \prod_i \left( \lambda_i \sqrt{\frac{\hbar \pi}{m \omega_i}} \right) \right] e^{\sum_i \frac{m \omega_i}{\hbar} \left( -\frac{x_i^2}{2 \lambda_i^2} + i \beta_i x_i^2 \right)} . \quad (4)$$

This is appropriate for the experiments performed at JILA which are not in the Thomas-Fermi limit. In fact, the ratio of  $\hbar\bar{\omega}$  and  $T^{2B}n_0$  is about one halve, where  $\bar{\omega} = (\omega_r^2\omega_z)^{1/3}$ . However, choosing the condensate wave function to be a gaussian is known to give quantitatively good results for the condensate modes, even in the Thomas-Fermi limit where the ground state is well approximated by an inverted parabola. Therefore, we expect the above *ansatz* to be also at least qualitatively applicable to the MIT experiments.

Second, to find an appropriate variational function for the quasiparticle distribution function, we take the distribution function to be an approximate Maxwell distribution for a gas of non-interacting bosons. To describe the monopole and quadrupole modes we introduce three scaling paramaters  $\{\alpha_i\}$  in the directions  $\{x_i\}$  respectively, where  $i = \{1, 2, 3\}$ . Hence,

$$F(\mathbf{k}, \mathbf{x}, t) = N' \left[ \prod_i (\beta \hbar \omega_i C_i) \right] e^{-\beta \sum_i C_i \left[ \frac{\hbar^2 \alpha_i^2}{2m} \left( k_i - \frac{m}{\hbar} \frac{\dot{\alpha}_i}{\alpha_i} x_i \right)^2 + \frac{m \omega_i^2}{2} \frac{x_i^2}{\alpha_i^2} \right]}. \quad (5)$$

This results in a time dependent density profile  $n'(\{x_i/\alpha_i(t)\})$ . For such a profile, the local current should be given by  $\hbar \langle k_i \rangle(\mathbf{x}, t)/m = \dot{\alpha}_i(t) x_i/\alpha_i(t)$ , which is correctly reproduced by our trial function. Furthermore, the equilibrium values for  $\alpha_i$  are in general different for  $i = \{1, 2, 3\}$  and larger than one, due to the interactions. This will cause the equilibrium density profile to broaden, but unfortunately also cause the distribution in momentum space to become anisotropic, which is incorrect because the mean-field interactions have no momentum dependence. The constant overall factors  $\{C_i\}$  are introduced to compensate for this by choosing the value of  $C_i$  equal to  $1/\alpha_{i,e}^2$ , where  $\alpha_{i,e}$  denotes the equilibrium value of  $\alpha_i$ .

We can now find evolution equations for the variational parameters  $\{\beta_i, \lambda_i, \alpha_i\}$  by minimizing the lagrangian for the nonlinear Schrödinger equation and writing the Boltzmann equation as a set of moment equations, that truncates because of the variational *ansatz*. Our main result, the equations for  $\{\beta_i\}, \{\lambda_i\}$ , and  $\{\alpha_i\}$ , therefore reads

$$\beta_i = \frac{1}{2\omega_i} \frac{\dot{\lambda}_i}{\lambda_i}, \quad (6)$$

$$\begin{aligned} \frac{\ddot{\lambda}_i}{\omega_i^2} + \lambda_i &= \frac{1}{\lambda_i^3} + \sqrt{\frac{2}{\pi}} \frac{N_0 a}{l_i} \left( \frac{l_i}{\bar{l}} \right)^3 \left[ \prod_j \frac{1}{\lambda_j} \right] \frac{1}{\lambda_i} + \\ &\quad \frac{4N'}{\pi\sqrt{2\pi}} \left( \frac{\Lambda_{th}}{\bar{l}} \right)^5 \frac{a}{\bar{l}} \left[ \prod_j \frac{1}{\beta \hbar \omega_j \lambda_j^2 + 2\alpha_j^2 \alpha_{j,e}^2} \right] \frac{\lambda_i}{\beta \hbar \omega_i \lambda_i^2 + 2\alpha_i^2 \alpha_{i,e}^2} \end{aligned} \quad (7)$$

$$\begin{aligned} \frac{\ddot{\alpha}_i}{\omega_i^2} + \alpha_i &= \frac{1}{\alpha_i^3} + \frac{N'}{8\pi^3\sqrt{2}} \left( \frac{\Lambda_{th}}{\bar{l}} \right)^5 \frac{a}{\bar{l}} \left[ \prod_j \frac{1}{\alpha_j \alpha_{j,e}} \right] \frac{1}{\alpha_i \alpha_{i,e}^2} + \\ &\quad \frac{4N_0}{\pi^3\sqrt{2}} \left( \frac{\Lambda_{th}}{\bar{l}} \right)^5 \frac{a}{\bar{l}} \left[ \prod_j \frac{1}{\beta \hbar \omega_j \lambda_j^2 + 2\alpha_j^2 \alpha_{j,e}^2} \right] \frac{\alpha_i}{\beta \hbar \omega_i \lambda_i^2 + 2\alpha_i^2 \alpha_{i,e}^2}. \end{aligned} \quad (8)$$

Here,  $\Lambda_{th}$  denotes the thermal wavelength  $(2\pi\hbar/mk_BT)^{1/2}$ ,  $\bar{l}$  is equal to  $(l_r^2 l_z)^{1/3}$ ,  $l_i$  denotes the harmonic oscillator length  $(\hbar/m\omega_i)^{1/2}$ , and  $N_0$  and  $N'$  denote the total number of condensate and noncondensed atoms, respectively.

To determine the spectrum of the low-lying collective excitations as a function of  $T/T_c$ , we take the critical temperature and the number of condensate and noncondensed atoms to be given by the same expressions as those for an ideal gas in a trap, i.e.  $k_B T_c = \hbar\bar{\omega}(N/1.202)^{1/3}$ ,  $N_0 = N [1 - (T/T_c)^3]$ , and  $N' = N (T/T_c)^3$ . In addition, we have included the effects of evaporative cooling, by approximating the dependence of the total number of particle  $N$  on  $T/T_c$  as found in the JILA experiment by a second order polynomial. We then find the fixed point of equations (6),(7) and (8) for each temperature and calculate the eigenvectors and their eigenvalues by linearizing around the fixed point. The results are plotted in figure 1.

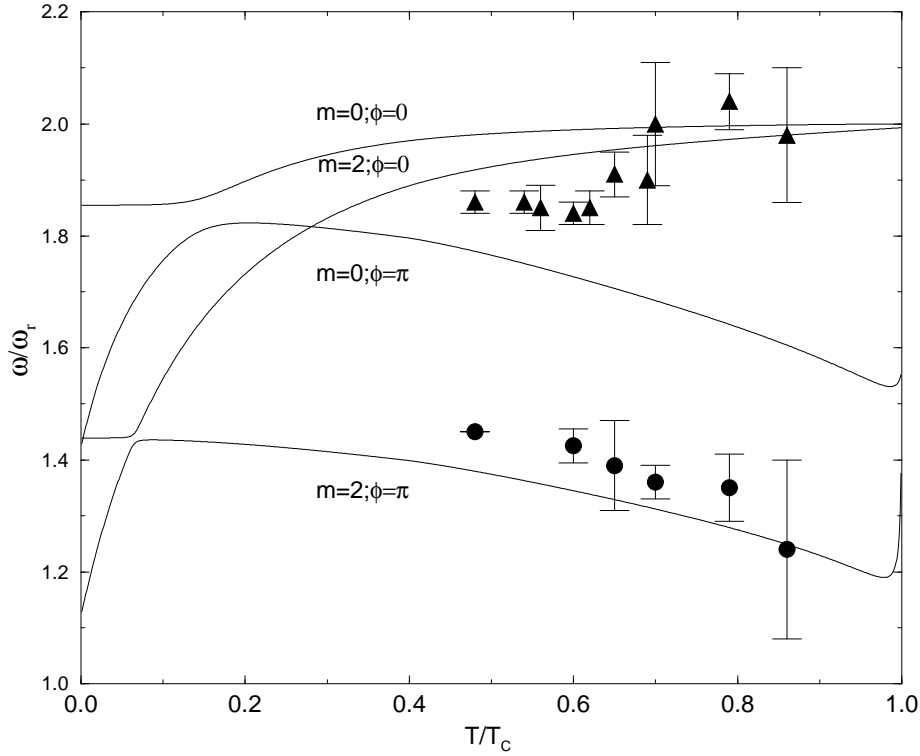


Figure 1: The low-lying  $m = 0, 2$  collisionless modes as a function of  $T/T_c$ , where  $\phi$  denotes the relative phase of density profiles of the condensed and noncondensed atoms. Also included are the experimental results for the  $m=0$  (triangles) and  $m=2$  (circles) modes found in the JILA experiment [9]

## 4 DISCUSSION AND CONCLUSION

Instead of the usual two modes, found when solving the Gross-Pitaevskii equation or the nonlinear Schrödinger equation, there are now four modes. This

should not come as a surprise, since we are essentially dealing with two gas clouds. Without an interaction between these clouds, there would be a monopole  $m = 0$  and a quadrupole  $m = 2$  mode for both the condensate and the noncondensed cloud. If we turn on the interaction these modes get coupled, resulting in four modes where the condensate and the thermal cloud move either in or out of phase. These are the collisionless analogues of the hydrodynamic first and second sound modes [14, 15]. Most importantly, we see from figure 1 that the temperature dependence of the  $m = 2$  mode is in good agreement with the JILA experiment, in contrast with the previous approaches which do not take into account the dynamics of the noncondensed cloud. In addition, the  $m = 0$  mode does have the correct non-interacting limit near  $T_c$ , but the experimental data drops to the zero-temperature limit  $(10/3)^{1/2} \omega_r$  at a higher temperature than our theoretical curve, which nevertheless shows qualitatively the same behavior. We therefore believe that the avoided crossing between the in and out of phase monopole modes that causes this behavior, might be the reason for the strong temperature dependence of the  $m = 0$  mode found experimentally. The reason that here the quantitative agreement between our theory and the experiments is not that good, might be either the use of the Hartree-Fock approximation, or of the gaussian approximation for the condensate wave function. Work to improve on these approximations is in progress. Finally, we note that it might be possible to observe also the other two modes experimentally. Whether this is possible depends on the overlap of these modes with the applied perturbation, and on the damping of the modes, which we have neglected thusfar. However, in principle the collisionless Boltzmann equation also contains Landau damping. Moreover, by including the collision term we should be able to describe collisional damping in the same way as Kavoulakis, Pethick and Smith [16].

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